Frequency Estimation from two DFT Bins

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In a recent thread in comp.dsp [1] Cedron Dawg presented an exact formula for extracting the frequency of a pure tone from three DFT bins [2]. The possibility of using fewer bins remained an open question, also whether or not there are other three-bin formulas.

Inspired by that thread I was able to derive an exact expression for the frequency based on values from only two bins. It turns out that that expression is not unique: it offers a degree of freedom which may be used to advantage in the presence of noise.

Consider a frame of real valued samples $x_0, x_1, \ldots, x_{N-1}$ which represent a pure tone with some frequency, amplitude, and phase. We may use the center position $n_c = (N-1)/2$ to write

$$x_n = A\cos[(n-n_c)\omega] + B\sin[(n-n_c)\omega], \tag{1}$$

where ω denotes a normalized angular frequency in the range $0 \le \omega \le \pi$ with $\omega = \pi$ representing the Nyquist frequency. A and B are real valued constants to allow for an arbitrary amplitude $\sqrt{A^2+B^2}$ and phase $\arctan(B/A)$.

The DFT may be written in a form centered around n_c,

$$X_k = e^{2\pi i k n_c/N} \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N}, \tag{2}$$

where the sum on the right hand side represents the familiar DFT and the phase factor in front provides for the symmetry with respect to n_c.

Inserting (1) into (2) yields, after some algebra,

$$X_k = (-1)^k \frac{2\sin(N\omega/2)}{\cos\beta_k - \cos\omega} \left[A\sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\beta_k}{2}\right) + iB\cos\left(\frac{\omega}{2}\right)\sin\left(\frac{\beta_k}{2}\right)\right], \tag{3}$$

where we have introduced the abbreviation $\beta_k = 2\pi k/N$. We outline the intermediate steps at the end of this note.

Suppose the set of X_k is known for all $k = 0, 1, \ldots, N-1$, and the task is to extract the frequency ω, and possibly A and B. To this end we define the quantities

$$u_k = (-1)^k \operatorname{Re}(X_k)/\cos(\beta_k/2), \quad k \neq N/2. \tag{4}$$

It is easy to see that the following relation holds for any $k \neq j$:

$$\frac{u_j}{u_k} = \frac{\cos\beta_k - \cos\omega}{\cos\beta_j - \cos\omega}. \tag{5}$$

Eq.(5) may be solved for ω to yield

$$\omega = \arccos\left(\frac{u_j\cos\beta_j - u_k\cos\beta_k}{u_j - u_k}\right) \tag{6}$$

So far so good: equation (6) provides a formula for the frequency ω using information from the two bins j and k. Note that j and k need not denote adjacent bins, although it is reasonable to choose the two bins with maximum modulus, which, in the case of a pure tone spectrum, will be adjacent bins. Also note that these two bins will enclose ω, and that the corresponding u_j and u_k will have oposite signs.

Eq.(5) will break down for $A = 0$, however. In that case we may define

$$v_k = (-1)^k \operatorname{Im}(X_k)/\sin(\beta_k/2), \quad k \neq 0, \tag{7}$$

and obtain similar formulas as in eqs.(5) and (6). Indeed, any linear combination

$$w_k = a u_k + b v_k \tag{8}$$

will do.

This degree of freedom is not so surprising given that two bins hold two complex valued numbers (i.e. four independent values) to determine three unknowns ω, A and B. By the same token, one would expect Cedron's three-bin formula to be only one among many others. Clearly, we can use this freedom to our advantage and make an optimum choice of e.g. a and b for best signal to noise ratio. Although it is not obvious what the best choice would be, setting

$$\begin{aligned} a &= \cos(\beta_m/2)\operatorname{Re}(X_m) \\ b &= \sin(\beta_m/2)\operatorname{Im}(X_m) \end{aligned} \tag{9}$$

with m denoting the peak bin optimizes the SNR of w_m. Hence that choice is expected to provide nearly optimum accuracy.

It would seem that because only two bins are used, much information is actually thrown away - information that could be used if the background characteristics are known (white noise being the simplest case). On the other hand, for unknown background one might argue that a two-bin method provides maximum locality hence minimum interference. I have tested the scheme only briefly in a real-world scenario with noise or another interfering tone. Results with w_k according to eq.(9) are indeed better than with either u_k or v_k plugged into eq.(6).

Derivation of Eq.(3)

We may write eq.(1) in the form

$$x_n = \frac{A - iB}{2} e^{i\omega(n-n_c)} + \frac{A + iB}{2} e^{-i\omega(n-n_c)}. \tag{10}$$

The DFT of the exponentials in eq.(10) is

$$\sum_{n=0}^{N-1} e^{-i(\beta_k \pm \omega)(n-n_c)} = \frac{\sin\left[N(\beta_k \pm \omega)2\right]}{\sin[(\beta_k \pm \omega)/2]} = \frac{(-1)^k \sin(N\omega/2)}{\sin[(\omega \pm \beta_k)/2]}. \tag{11}$$

Hence we obtain the following expression for the real part:

$$\mathrm{Re}(X_k) = \frac{(-1)^k A}{2} \sin\left(\frac{N\omega}{2}\right) \left\{ \frac{1}{\sin[(\omega + \beta_k)/2]} + \frac{1}{\sin[(\omega - \beta_k)/2]} \right\} \tag{12}$$

The term in curly brackets in eq.(12) may be simplified using trigonometric identities, resulting in

$$4\,\frac{\sin(\omega/2)\cos(\beta_k/2)}{\cos\beta_k - \cos\omega}. \tag{13}$$

The imaginary part $\mathrm{Im}(X_k)$ may be treated in a similar way. Collecting all bits and pieces, we obtain the result in eq.(3).

Derivation of Optimum a and b

If we assume additive white noise, then the Fourier components will fluctuate with some variance σ^2, regardless of k, and the variance will be equal for the real and imaginary parts:

$$\mathrm{Var}[\mathrm{Re}(X_k)] = \mathrm{Var}[\mathrm{Im}(X_k)] = \sigma^2. \tag{14}$$

Hence the variances of u_k and v_k will be

$$\mathrm{Var}(u_k) = \frac{\sigma^2}{\cos^2(\beta_k/2)}, \quad \mathrm{Var}(v_k) = \frac{\sigma^2}{\sin^2(\beta_k/2)}, \tag{15}$$

which yields the variance of w_k,

$$\mathrm{Var}(w_k) = \left[\frac{a^2}{\cos^2(\beta_k/2)} + \frac{b^2}{\sin^2(\beta_k/2)}\right]\sigma^2 \tag{16}$$

Now the *relative mean standard deviation* of w_k is $\sqrt{\mathrm{Var}(w_k)}/|w_k|$. We want the relative error of w_k to be small because we are taking ratios. Minimizing the above expression (or easier, its square) results in a condition for a and b,

$$\frac{a}{b} = \frac{\cos(\beta_k/2)\mathrm{Re}(X_k)}{\sin(\beta_k/2)\mathrm{Im}(X_k)}. \tag{17}$$

Minimum Variance Estimate

The information from two bins leads to an overdetermined system of equations for ω. With the abreviations $\mu = \cos\omega$, and $\mu_k = \cos\beta_k$, these equations are:

$$\begin{aligned} U(\mu) &:= (u_j - u_k)\mu - u_j\mu_j + u_k\mu_k = 0 \\ V(\mu) &:= (v_j - v_k)\mu - v_j\mu_j + v_k\mu_k = 0 \end{aligned} \tag{18}$$

In general, the two equations cannot be solved simultaneously. The best one can do is to minimize a suitable error. One may square each equation and minimize their weighted sum. The optimum weights are the inverse variances of each term. Observe that for adjacent j and k at the peak,

$$\mathrm{Var}(U) \propto \sigma^2/\cos^2(\beta_m/2) \quad \mathrm{Var}(V) \propto \sigma^2/\sin^2(\beta_m/2) \tag{19}$$

where the index m denotes the peak. Hence the optimum cost function is

$$\cos^2(\beta_m/2)\,U(\mu)^2 + \sin^2(\beta_m/2)\,V(\mu)^2 = \min. \tag{20}$$

Taking the derivative with respect to μ and equating to zero, we arrive at the expression

$$\begin{aligned} \mu = \cos\omega &= \frac{\cos^2(\beta_m/2)(u_j - u_k)^2\mu_u + \sin^2(\beta_m/2)(v_j - v_k)^2\mu_v}{\cos^2(\beta_m/2)(u_j - u_k)^2 + \sin^2(\beta_m/2)(v_j - v_k)^2} \\ &\approx \frac{[\mathrm{Re}(X_j + X_k)]^2\mu_u + [\mathrm{Im}(X_j + X_k)]^2\mu_v}{|X_j + X_k|^2} \end{aligned} \tag{21}$$

where μ_u and μ_v are the solutions of $U(\mu) = 0$ and $V(\mu) = 0$, respectively. This result coincides with the one found in the previous section if we substitute in eq. (9) $u_m \approx |u_j - u_k|/2$ and $v_m \approx |v_j - v_k|/2$. Recall that u_j and u_k have opposite signs, likewise for v_j and v_k.

Formulas in Terms of Ordinary DFT

For easier application of the results we provide the main expressions in terms of the standard DFT which we denote by Z_k,

$$Z_k = \sum_{n=0}^{N-1} x_n e^{-i\beta_k n}. \tag{22}$$

The relation between the centered and the standard DFT is

$$\begin{aligned}(-1)^k \mathrm{Re}(X_k) &= \cos(\beta_k/2)\,\mathrm{Re}(Z_k) + \sin(\beta_k/2)\,\mathrm{Im}(Z_k)\\ (-1)^k \mathrm{Im}(X_k) &= -\sin(\beta_k/2)\,\mathrm{Re}(Z_k) + \cos(\beta_k/2)\,\mathrm{Im}(Z_k),\end{aligned} \tag{23}$$

The auxiliary quantities u_k and v_k become

$$\begin{aligned}u_k &= \mathrm{Re}(Z_k) + \tan(\beta_k/2)\,\mathrm{Im}(Z_k)\\ v_k &= -\mathrm{Re}(Z_k) + \cot(\beta_k/2)\,\mathrm{Im}(Z_k),\end{aligned} \tag{24}$$

and the optimum choice for a and b is

$$\frac{a}{b} = \frac{\cot(\beta_m/2)\,\mathrm{Re}(Z_m) + \mathrm{Im}(Z_m)}{-\tan(\beta_m/2)\,\mathrm{Re}(Z_m) + \mathrm{Im}(Z_m)}. \tag{25}$$

Complex Tone

As an alternative to eq.(1), consider a complex tone

$$x_n = C e^{i(n-n_c)\omega}, \tag{26}$$

where C represents a complex amplitude. The DFT according to eq.(2) is

$$X_k = (-1)^k C \frac{\sin(N\omega/2)}{\sin[(\beta_k - \omega)/2]}. \tag{27}$$

To derive a two-bin formula for a complex tone, we form the ratio

$$\frac{(-1)^j X_j}{(-1)^k X_k} = \frac{\sin[(\beta_k - \omega)/2]}{\sin[(\beta_j - \omega)/2]} = \frac{\cos(\omega/2)\sin(\beta_k/2) - \sin(\omega/2)\cos(\beta_k/2)}{\cos(\omega/2)\sin(\beta_j/2) - \sin(\omega/2)\cos(\beta_j/2)} \tag{28}$$

which is easily solved for

$$\tan(\omega/2) = \frac{(-1)^j X_j \sin(\beta_j/2) - (-1)^k X_k \sin(\beta_k/2)}{(-1)^j X_j \cos(\beta_j/2) - (-1)^k X_k \cos(\beta_k/2)}. \tag{29}$$

For adjacent bins $j = k + 1$ this expression becomes

$$\tan(\omega/2) = \frac{X_{k+1} \sin(\beta_{k+1}/2) + X_k \sin(\beta_k/2)}{X_{k+1} \cos(\beta_{k+1}/2) + X_k \cos(\beta_k/2)}. \tag{30}$$

References

[1] http://www.dsprelated.com/showthread/comp.dsp/278049-1.php

[2] http://www.dsprelated.com/showarticle/773.php