Complex Waveshapers

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Abstract

Complex waveshapers take a sine and cosine wave pair as input and deliver two output waves F and G with different shapes but identical harmonic intensities, which follow from a Taylor expansion of the shaper function. F and G form a Hilbert pair, hence their partial frequencies may be shifted to produce sounds with other harmonic relationships. A number of examples are presented.

1 Introduction

Waveshaping [1] is a technique used in audio processing to alter the harmonic content and possibly also the dynamic range of a given signal. In the simplest case, a sine wave is transformed to a more complex waveform with some harmonic spectrum.

Waveshaping functions may be of arbitrary form in principle. The most common ones display linear behavior for small signal amplitudes and then gradually saturate as the amplitude increases. However, in some synths a particular timbre is achieved with a shaping function which folds back at some point [2] .

If we allow for simultaneous sine and cosine input, we may enlarge the variety of possible waveforms. Moreover, if the shaping function is complex and analytic, the real and imaginary outputs will form a Hilbert pair of waveforms with different shapes but identical harmonic intensities.

2 Harmonic series and generating functions

Consider a complex analytic function $H(z)$ with a known Taylor series expansion for z within the region of convergence,

$$H(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{1}$$

If we substitute $re^{i\omega t}$ for z, then obviously

$$H(re^{i\omega t}) = \sum_{n=0}^{\infty} b_n r^n e^{in\omega t}.$$

Using Euler's identity $e^{i\phi} = \cos\phi + i\sin\phi$ for each term $e^{in\omega t}$ in eq.(1), we can evaluate the real and imaginary parts, respectively, to obtain

$$\mathrm{Re}(H) = \sum_{n=0}^{\infty} b_n r^n \cos(n\omega t), \quad \mathrm{Im}(H) = \sum_{n=0}^{\infty} b_n r^n \sin(n\omega t).$$

Hence the Taylor expansion coefficients b_n of the generating function H are essentially the harmonic series coefficients.

In musical applications we usually do not want a DC bias, so we need to get rid of the term with $n = 0$. Furthermore, we wish to normalize the outputs so that the amplitude of the fundamental is unity. To this end, define functions F and G as

$$F = \frac{\mathrm{Re}(H) - b_0}{b_1 r}, \quad G = \frac{\mathrm{Im}(H)}{b_1 r}. \tag{2}$$

With this definition, we obtain the following harmonic series expansions

$$F = \sum_{n=1}^{\infty} a_n \cos(n\omega t), \quad G = \sum_{n=1}^{\infty} a_n \sin(n\omega t), \tag{3}$$

where the partial intensities are given by

$$a_0 = 0, \quad a_1 = 1, \quad a_n = b_n r^{n-1}/b_1, \quad n = 2, 3, 4, \ldots \tag{4}$$

Given a complex function $H(z)$, equation (2) provides a scheme to generate waveforms with a harmonic spectrum given by the Taylor series coefficients of $H(z)$, by applying functions F or G to the sine and cosine outputs of a quadrature oscillator. In this sense, F and G act as the real and imaginary parts of a complex waveshaper. In the following sections we will examine some simple, closed-form waveshapers.

3 Examples of waveshapers

3.1 Geometric series

A simple generating function is $H(z) = 1/(1 - z)$, which has a geometric series expansion,

$$H(z) = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n. \tag{5}$$

i.e., $b_n = 1$ for all n. $H(z)$ may be written in terms of its real and imaginary parts,

$$H(z) = \frac{1 - x + iy}{(1 - x)^2 + y^2}, \quad \text{where } z = x + iy.$$

Let r be a real number with $|r| < 1$. Substitute $re^{i\omega t}$ for z, then

$$H = \frac{1}{1 - re^{i\omega t}} = \frac{1 - rc + irs}{1 + r^2 - 2rc}$$

where we have used abbreviations $c = \cos\omega t$ and $s = \sin\omega t$. The expression on the right hand side makes it easy to identify the real and imaginary part, respectively,

$$\mathrm{Re}(H) = \frac{1 - rc}{1 + r^2 - 2rc},$$
$$\mathrm{Im}(H) = \frac{rs}{1 + r^2 - 2rc}.$$

Hence, if we drop the DC term and rescale, we arrive at the waveforms

$$F = \frac{c - r}{1 + r^2 - 2rc}, \quad \text{bounds } \frac{-1}{r \pm 1}, \tag{6}$$
$$G = \frac{s}{1 + r^2 - 2rc}, \quad \text{bounds } \frac{\pm 1}{1 - r^2}. \tag{7}$$

and the partials amplitudes are

$$a_n = r^{n-1}, \quad n = 1, 2, 3, \ldots \tag{8}$$

The fundamental frequency has unity amplitude, and higher harmonics fall off geometrically as r^{n-1}. For $r = 0$ there are no higher harmonics. For r approaching unity, $r = 1 - \epsilon$ with $\epsilon \ll 1$, partial amplitudes remain nearly constant up to $n \approx 1/\epsilon$ and then fall off for higher n. The resulting waveforms are simple rational functions. The lower and upper bounds for F are $-1/(1 + r)$ and $1/(1 - r)$, respectively, and $\pm 1/(1 - r^2)$ for G. Obviously the sine series results in a symmetric waveform F with a smaller crest factor than the cosine series waveform G.

Figure 1 shows two cycles of waveforms F and G, respectively, for selected values $r = 0$, 0.5, 0.7, and 0.8. Figure 2 depicts the corresponding partial amplitudes.

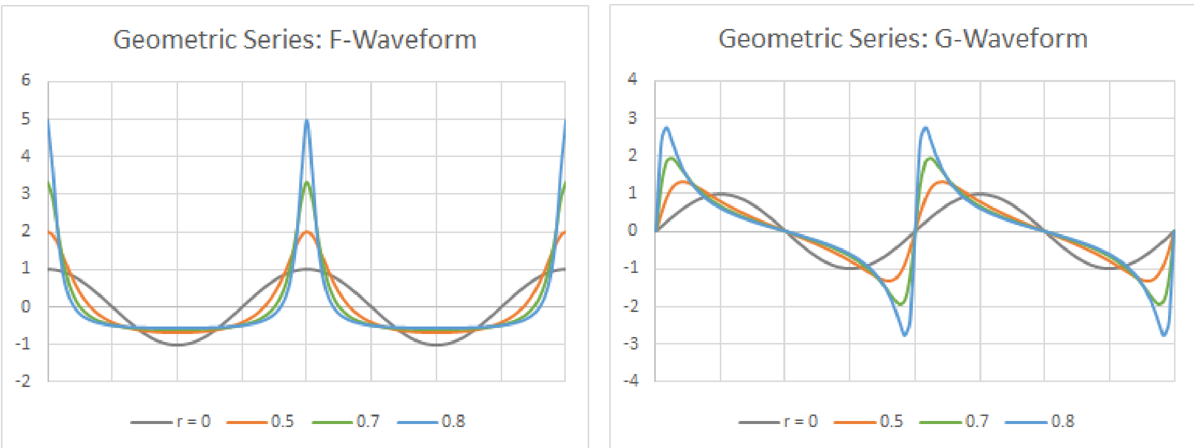


Figure 1: Geometric series waveforms, eqs.(6) and (7).

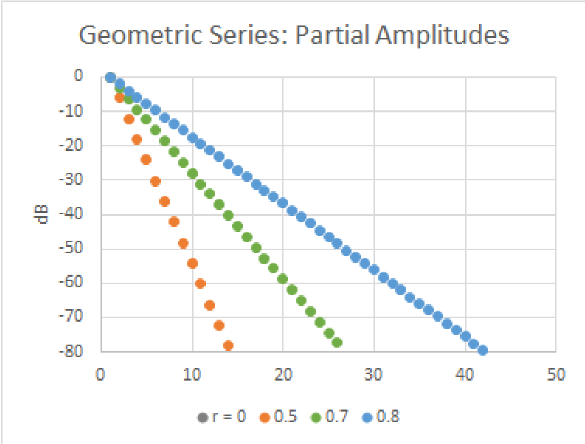


Figure 2: Geometric series partial amplitudes, eq.(8).

3.2 Exponential

Take $H(z) = e^z$ as the generating function, with the well-known series expansion

$$H(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x(\cos y + i \sin y). \tag{9}$$

Substitute $re^{i\omega t}$ for z, with r being a real number without any particular bound. Performing the same manipulations as in the previous section, we arrive at

$$F = \frac{\cos(rs)\exp(rc) - 1}{r}, \tag{10}$$

$$G = \frac{\sin(rs)\exp(rc)}{r}. \tag{11}$$

Unfortunately, there is no closed form for the bounds. The partials amplitudes are

$$a_n = \frac{r^{n-1}}{n!}, \quad n = 1, 2, 3, \ldots \tag{12}$$

Figure 3 shows F and G waveforms for $r = 0$, 1, 2, and 3. Corresponding partial amplitudes are shown in figure 4. For $|r| > 1$ the first few partials a_n are stronger than the fundamental a_1 but then fall of very quickly (note the abscissa scale in figure 4).

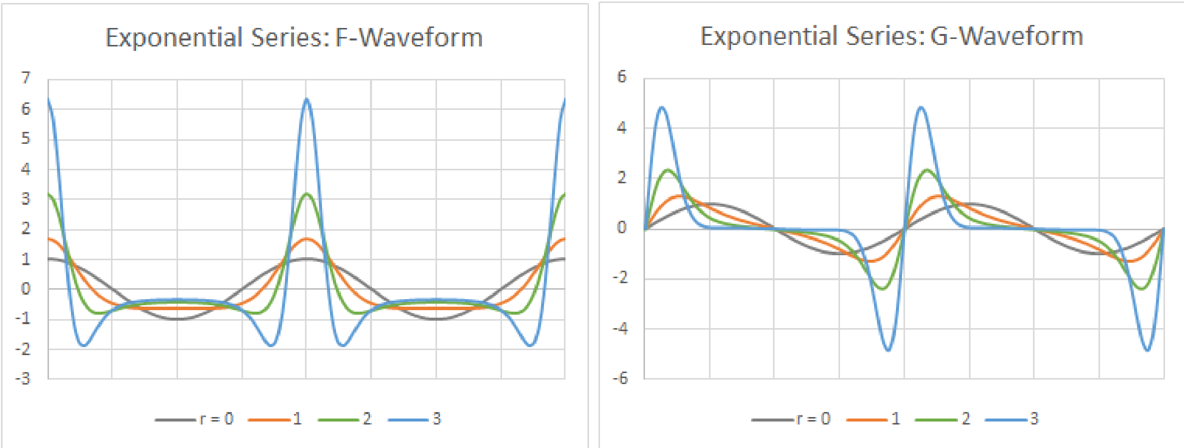


Figure 3: Exponential series waveforms, eqs.(10) and (11).

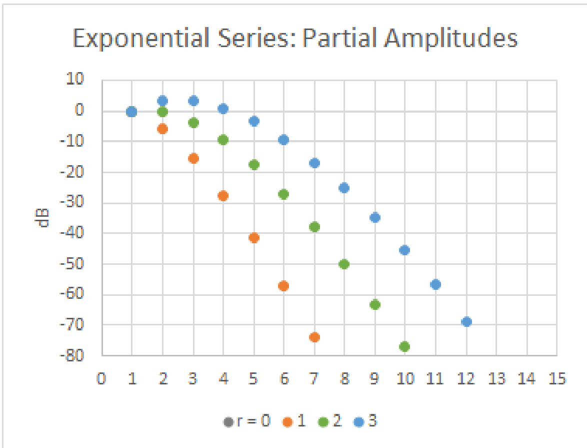


Figure 4: Exponential series partial amplitudes, eq.(12).

3.3 Logarithm

Consider the generating function [3, 4]

$$\begin{aligned} H(z) &= -\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \\ &= -\tfrac{1}{2}\ln\left((1-x)^2+y^2\right) + i\arctan\frac{y}{1-x}. \end{aligned} \tag{13}$$

Let r be a real number with $|r| < 1$ and perform the above algebraic steps to obtain

$$F = -\frac{\ln(1+r^2-2rc)}{2r}, \quad \text{bounds } \ln(1 \pm r)/r, \tag{14}$$

$$G = \frac{1}{r}\arctan\frac{rs}{1-rc}, \quad \text{bounds } \pm\arcsin(r)/r. \tag{15}$$

The partials amplitudes are

$$a_n = \frac{r^{n-1}}{n}. \tag{16}$$

Figures 5 and 6 show waveforms and partial amplitudes for selected values of r. Since $|r| < 1$, partials are always decreasing with increasing n. Note that the limiting case $r \to 1$ results in a $1/n$ spectrum. In that case, G represents a saw-tooth wave, refer to the right figure 5.

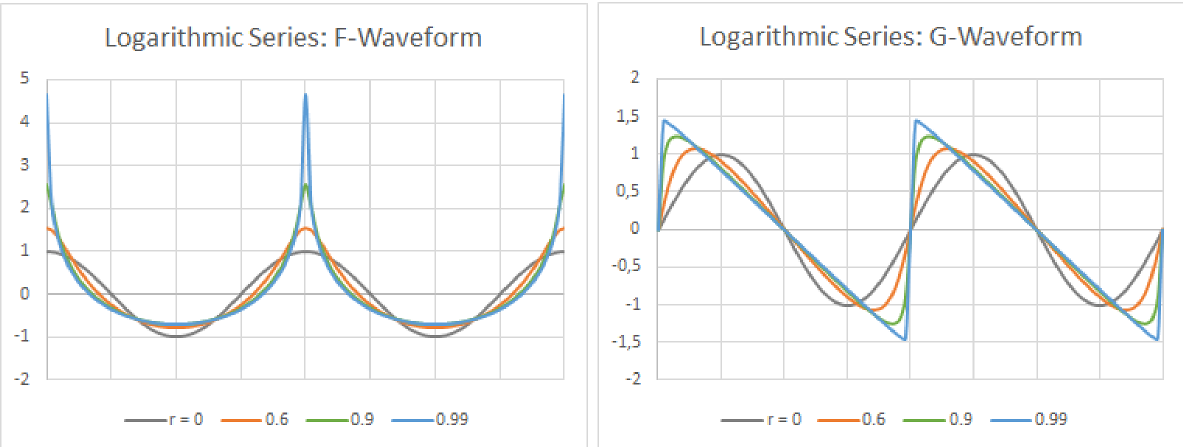


Figure 5: Logarithmic series waveforms, eqs.(14) and (15).

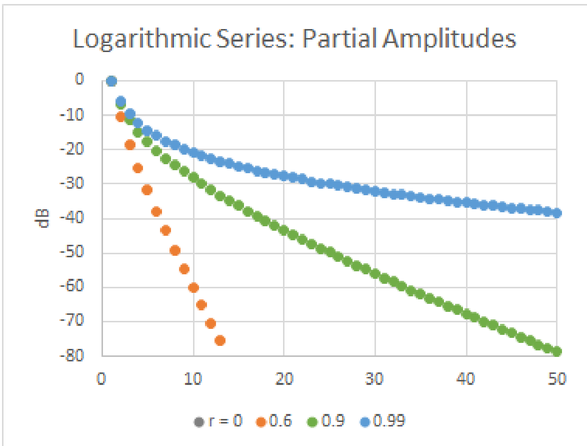


Figure 6: Logarithmic series partial amplitudes, eq.(16).

3.4 Power

Consider the generating function [3, 4]

$$\begin{aligned} H(z) &= (1+z)^\mu = \sum_{n=0}^{\infty} \binom{\mu}{n} z^n \\ &= \left((1+x)^2 + y^2\right)^{\frac{\mu}{2}} (\cos\nu + i \sin\nu), \quad \text{where } \nu = \mu \arctan\frac{y}{1+x}, \end{aligned} \tag{17}$$

with some real number μ. Let r be another real number with $|r| < 1$, then we get

$$F = \frac{(1 + r^2 + 2rc)^{\frac{\mu}{2}} \cos\nu - 1}{\mu r}, \tag{18}$$

$$G = \frac{(1 + r^2 + 2rc)^{\frac{\mu}{2}} \sin\nu}{\mu r}, \tag{19}$$

with

$$\nu = \mu \arctan \frac{rs}{1 + rc}.$$

The partials amplitudes are

$$a_n = \binom{\mu}{n} \frac{r^{n-1}}{\mu}, \qquad n = 1, 2, 3, \ldots \tag{20}$$

For integer and half-integer powers μ, alternative formulas may be derived, avoiding trigonometric functions. For positive integers $\mu = m$, the harmonic series terminates at the mth partial.

Figures 7–10 show examples for powers $\mu = -0.2$ and $\mu = 5$, and selected values for r. It is interesting to note that for large powers $|\mu| \gg 1$, the formulas approach exponential behavior, whereas for small powers $|\mu| \ll 1$, the results resemble logarithmic behavior.

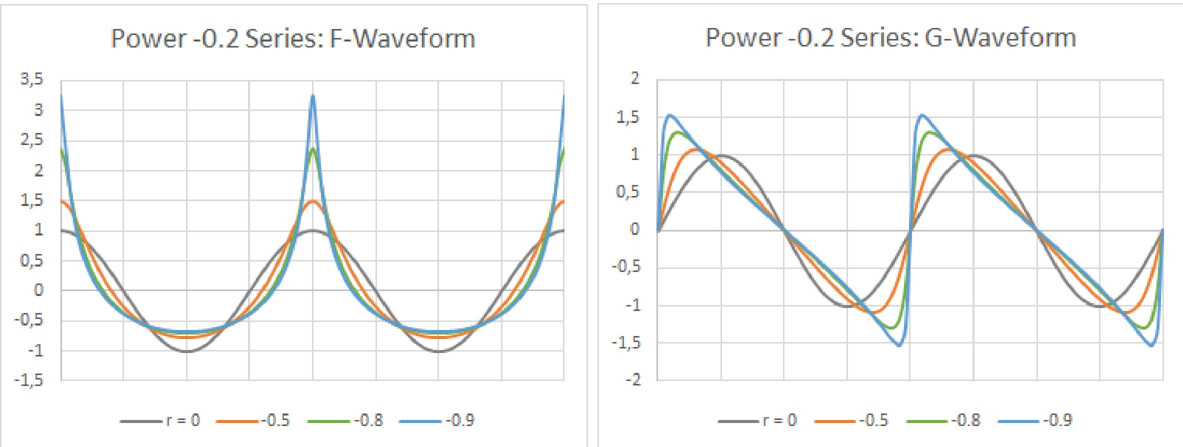


Figure 7: Power $mu = -0.2$ series waveforms, eqs.(18) and (19).

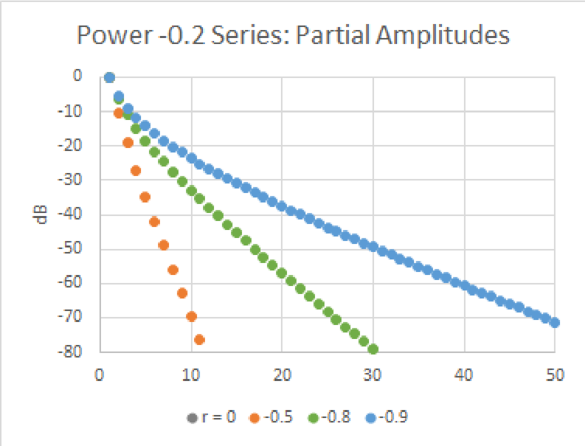
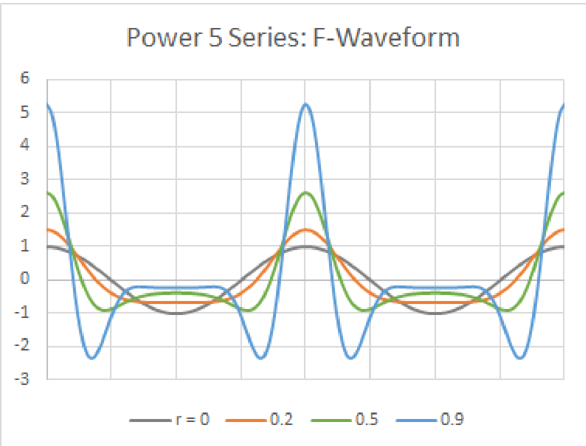


Figure 8: Power $mu = -0.2$ series partial amplitudes, eq.(20).



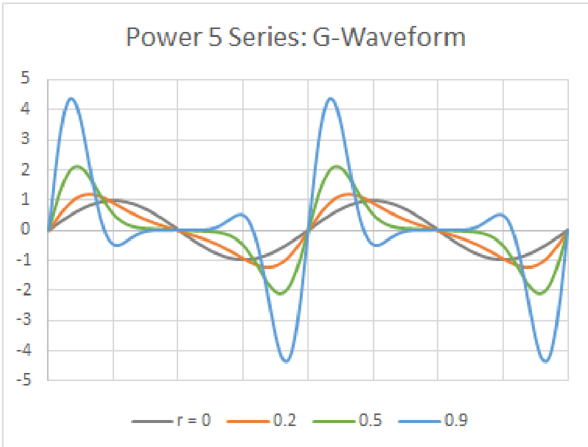


Figure 9: Power $mu = 5$ series waveforms, eqs.(18) and (19).

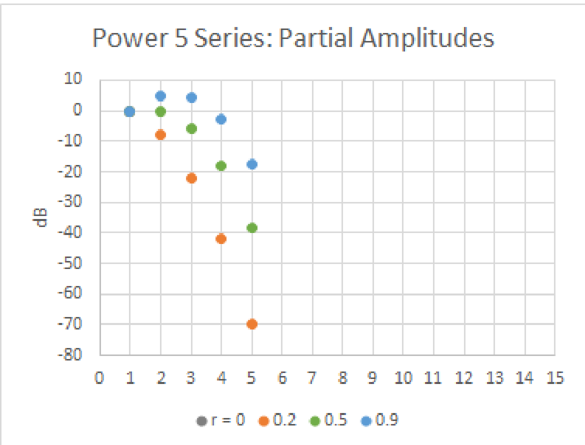


Figure 10: Power $mu = 5$ series partial amplitudes, eq.(20).

3.5 Sine

Consider the generating function [3, 4]

$$\begin{aligned} H(z) = \sin z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \tag{21}$$

The F and G waveforms are

$$F = \frac{\sin(rc)\cosh(rs)}{r}, \tag{22}$$

$$G = \frac{\cos(rc)\sinh(rs)}{r}, \quad \text{bounds } \pm \frac{\sinh r}{r}. \tag{23}$$

and the partial amplitudes are

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ r^{n-1}/n! & \text{if } n \text{ is odd} \end{cases} \tag{24}$$

Note that odd partials coincide with the exponential case, refer to eq.(12).

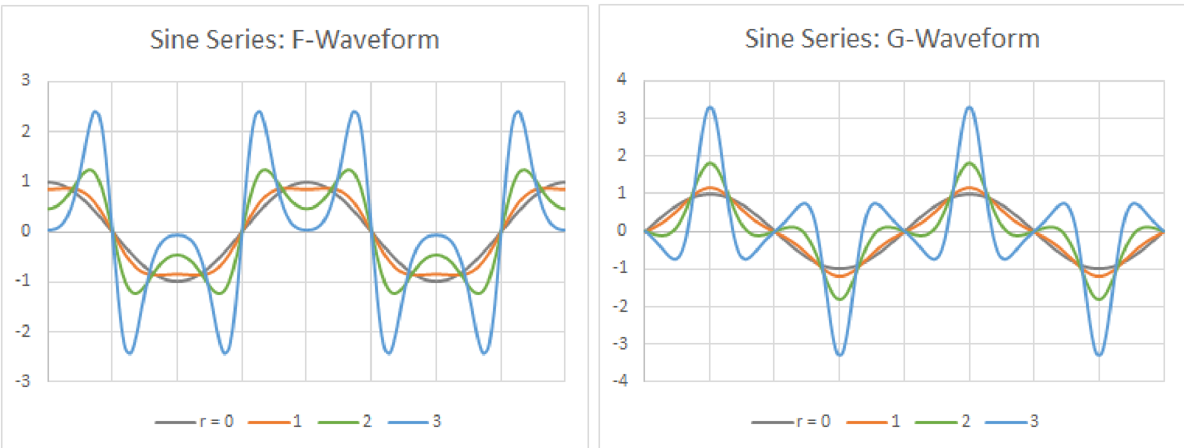


Figure 11: Sine series waveforms, eqs.(22) and (23).

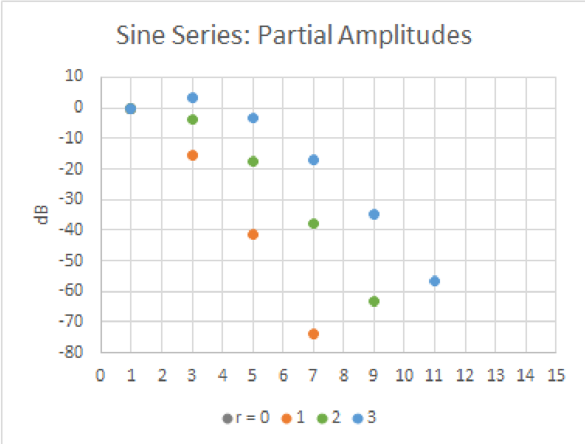


Figure 12: Sine series partial amplitudes, eq.(24).

3.6 Tangent

Consider the generating function [3, 4]

$$H(z) = \tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots, \quad |z| < \frac{\pi}{2}$$
$$= \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} \tag{25}$$

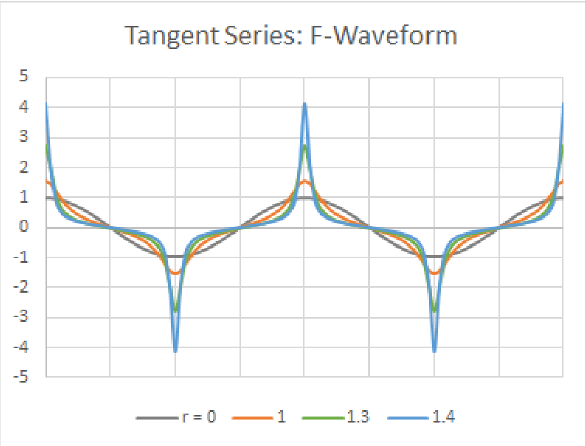
The F and G waveforms are

$$F = \frac{\sin 2rc}{r(\cos 2rc + \cosh 2rs)}, \quad \text{bounds } \pm\frac{\tan r}{r}, \tag{26}$$
$$G = \frac{\sinh 2rs}{r(\cos 2rc + \cosh 2rs)}. \tag{27}$$

and the partial amplitudes are

$$a_1 = 1, \quad a_3 = \frac{r^2}{3}, \quad a_5 = \frac{2r^4}{15}, \quad a_7 = \frac{17r^6}{315}, \quad \ldots \tag{28}$$

Note that even partials are zero. Odd partials are always decreasing for $|r| < \frac{\pi}{2}$. Figures 13 and 14 show examples.



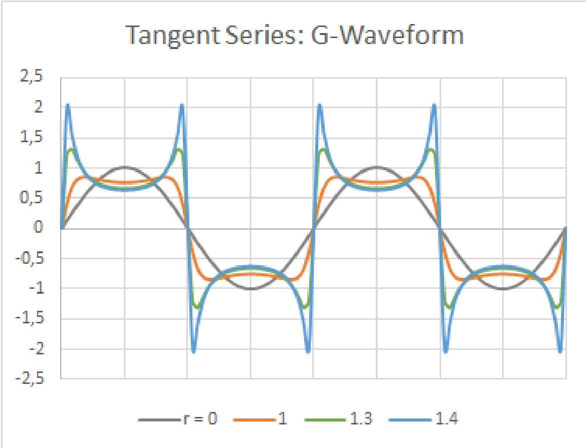


Figure 13: Tangent series waveforms, eqs.(26) and (27).

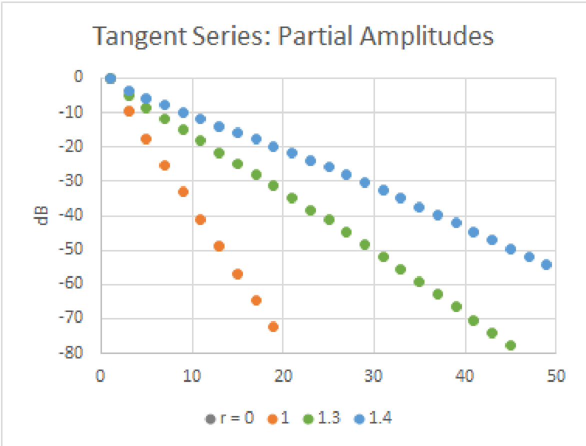


Figure 14: Tangent series partial amplitudes, eq.(28).

3.7 Arctangent

Consider the generating function [3, 4]

$$H(z) = \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)}, \quad |z| < 1$$

$$= \frac{1}{2} \arctan \frac{2x}{1 - x^2 - y^2} + \frac{i}{4} \ln \frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}. \qquad (29)$$

The F and G waveforms are

$$F = \frac{1}{2r} \arctan \frac{2rc}{1 - r^2}, \quad \text{bounds } \pm \frac{\arctan r}{r}, \qquad (30)$$

$$G = \frac{1}{4r} \ln \frac{1 + r^2 + 2rs}{1 + r^2 - 2rs}, \quad \text{bounds } \pm \frac{1}{2r} \ln \frac{1+r}{1-r}, \qquad (31)$$

with $|r| < 1$. The partial amplitudes are

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ r^{n-1}/n & \text{if } n \text{ is odd.} \end{cases} \tag{32}$$

Odd partials coincide with the logarithmic case, refer to eq.(16). For the limiting case $r \to 1$, the F waveform becomes rectangular, refer to left figure 15.

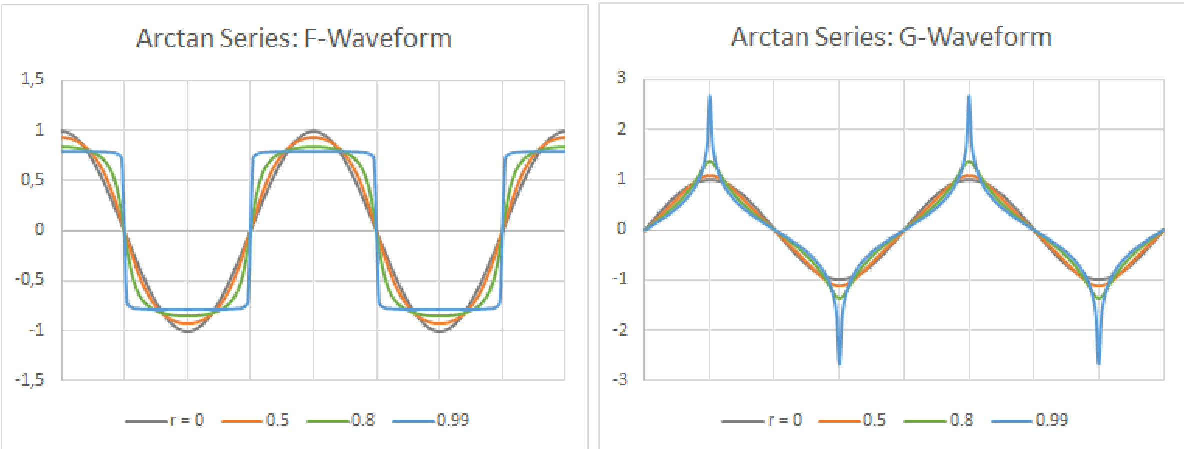


Figure 15: Arctangent series waveforms, eqs.(30) and (31).

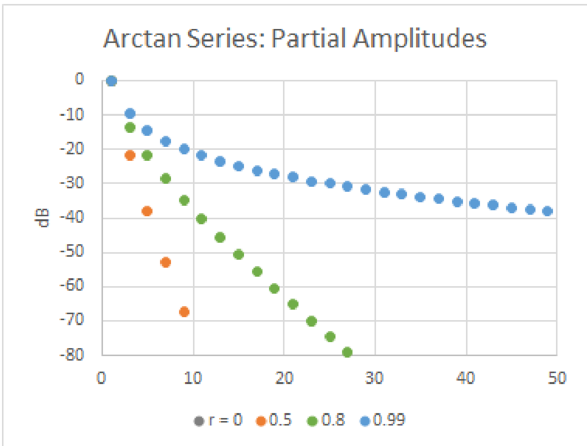


Figure 16: Arctangent series partial amplitudes, eq.(32).

4 Discussion

The few examples in section 3 already show some distinct features:

- increasing partial amplitudes with increasing parameter r (note that r is sometimes bound)

- monotonic or non-monotonic partial amplitudes
- finite or infinite number of partials
- even partials missing or present

Since the partial amplitudes determine the timbre of a wave sound, the method allows for a rich set of timbres. The r parameter is akin to a lowpass filter cutoff. Of the two waveforms F and G, we may choose the one with a favorable crest factor to avoid clipping, or the one which starts with zero amplitude to avoid clicks, etc. A linear combination is also possible.

4.1 Hilbert property

F and G are Hilbert pairs [5], hence partials can be shifted by means of a quadrature oscillator with frequency ω',

$$F \cos \omega' t - G \sin \omega' t = \sum_{n=1}^{\infty} a_n \cos \omega_n t \tag{33}$$

$$F \sin \omega' t + G \cos \omega' t = \sum_{n=1}^{\infty} a_n \sin \omega_n t \tag{34}$$

with $\omega_n = n\omega + \omega'$.

In general, the resulting sound will be anharmonic because the shifted partial frequencies ω_n are no longer at integer ratios. However, some special cases are notable exceptions. For instance shift down by $\omega' = -\frac{2}{3}\omega$ will create a new fundamental at $\omega_1 = \frac{1}{3}\omega$, first nonzero partial will be at $\omega_2 = \frac{4}{3}\omega = 4\omega_1$ (2 octaves up), next at $7\omega_1$ and so on. Such timbres with sparse partials are typical of instruments like vibraphone, glockenspiel, chimes, etc.

4.2 Aliasing suppression

Music production is entirely digital today, which means that signals are sampled and processed at discrete times t_n with a certain rate, the sampling rate. During processing, frequency components may arise which exceed the Nyquist limit, resulting in an unwanted artifact called aliasing. Various mitigation strategies exist, of which the most important are oversampling and ADAA [6, 7]

The idea of oversampling consists in processing at a higher sampling rate, hence pushing the Nyquist limit up for more headroom. Usually this involves first upsampling the signal at hand, then processing, and finally downsampling. In the present case we may skip the first step and generate the sin/cos inputs at a higher rate and feed them into H. No upsampling interpolation is needed.

The acronym ADAA stands for antiderivative anti-aliasing. The classical scheme involves four steps: (i) transform to continuous time by linear interpolation, (ii) carry out the processing, (iii) apply a boxcar filter of one sample width, and (iv) sample the result to get back to discrete time domain. Since waveforms F and G are known in continuous time domain, the first step may be omitted, and filtering can be applied directly. For instance a boxcar filter acting on H would be implemented as

$$\frac{\mathcal{H}(z_n) - \mathcal{H}(z_{n-1})}{z_n - z_{n-1}}, \tag{35}$$

where $z_n = re^{i\omega t_n}$ and $\mathcal{H}(z)$ is the (complex) antiderivative of $H(z)$,

$$\mathcal{H}(z) = \int^z H(z')dz'.$$

So to ADAA a complex wave shaper, use the expression (35) instead of $H(z_n)$.

References

[1] https://en.wikipedia.org/wiki/Waveshaper

[2] Fabián Esqueda, Henri Pöntynen, Vesa Välimäki, and Julian D. Parker, Virtual Analog Buchla 259 Wavefolder, Proceedings of the 20th International Conference on Digital Audio Effects (DAFx-17), Edinburgh, UK, September 5–9, 2017. https://www.dafx17.eca.ed.ac.uk/papers/DAFx17_paper_82.pdf

[3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions https://www.math.ubc.ca/~cbm/aands/

[4] Wolfram Alpha, Sums. https://www.wolframalpha.com/examples/mathematics/calculus-and-analysis/sums/

[5] Wikipedia, Hilbert Transform. https://en.wikipedia.org/wiki/Hilbert_transform

[6] Julian D. Parker, Vadim Zavalishin, Efflam Le Bivic, Reducing the Aliasing of Nonlinear Waveshaping Using Continuous-Time Convolution, Proceedings of the 19th International Conference on Digital Audio Effects (DAFx-16), Brno, Czech Republic, September 5–9, 2016. https://dafx.de/paper-archive/2016/dafxpapers/20-DAFx-16_paper_41-PN.pdf

[7] M. Vicanek, A note on Alias Suppression in Digital Distortion. https://vicanek.de/articles/AADistortion.pdf