

Matched Second Order Digital Filters

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1 Introduction

Second order sections are universal building blocks for digital filters. They are characterized by five coefficients, which determine the filter's transfer function, e.g. a lowpass, bandpass, bandstop etc. Filter design techniques exist to determine the coefficients for a specified filter characteristic, the most simple being the so-called bilinear transform. Other mappings include impulse invariance, matched Z-transform, Orfanidis method, Massberg, each addressing a particular aspect. In this note we present a novel method which provides simple, closed form expressions for the coefficients so the filter characteristic may be modulated with near sample rate updating.

The paper is organized as follows: in section 2 we briefly introduce the bi-quadratic filter (biquad) and establish notation. In section 3 we recall some common design methods. Sections 4 and 5 address the novel design method for biquads with explicit results for lowpass, bandpass, highpass and peaking EQ filters.

2 The Biquad

A biquad is a generic linear second order filter which may be described in terms of its action on a stream of samples by the following recurrence relation:

$$u_n = b_0x_n + b_1x_{n-1} + b_2x_{n-2} - a_1u_{n-1} - a_2u_{n-2} \quad (1)$$

Here, x_n denotes the current input sample, x_{n-1} is the previous sample etc., and u_n is the output sample. Such a filter is characterized by five coefficients b_0, b_1, b_2, a_2, a_2 . Unfortunately, there is no standard way of naming the coefficients, some authors have the a s and b s the other way round. You may also find different sign conventions, so always check the definitions before applying a ready-to-use formula.

Taking the Z-transform of the difference equation (1) yields the biquad transfer function

$$H(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}} \quad (2)$$

We recall that for a stable filter all poles of $H(z)$, if any, must lie inside the unit circle, which implies the following conditions for a_1 and a_2 :

$$|a_1| < 2, \quad |a_1| - 1 < a_2 < 1. \quad (3)$$

Inside this triangle the sign of the discriminant $a_2 - a_1^2/4$ determines whether the poles are real or complex conjugate:

$$\begin{aligned} a_2 > a_1^2/4 &\Rightarrow \text{complex conjugate poles} \\ a_2 < a_1^2/4 &\Rightarrow \text{real poles} \end{aligned} \quad (4)$$

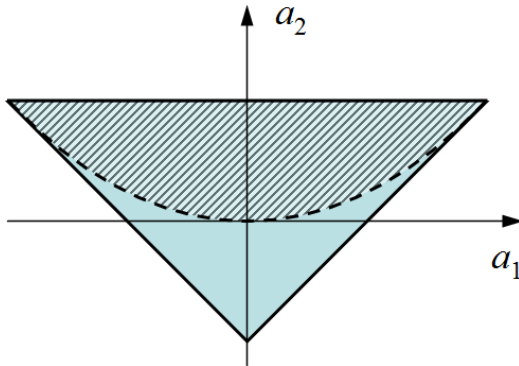


Figure 1: Stability bounds for biquad coefficients, eq. (3). Poles are complex conjugate above the dashed line, and are real below.

The rest of this paper deals with methods to determine the five coefficients to realize a filter for given specifications.

3 Filter Design Methods

Because filter theory was historically developed first for analog filters, many design methods start with an analog prototype and then employ some sort of mapping to the digital domain.

3.1 Bilinear Transform

The most common such mapping is accomplished by the bilinear transform (BLT):

$$w = \tan\left(\frac{\omega}{2}\right) = i \frac{1 - z^{-1}}{1 + z^{-1}} \quad (5)$$

The above equation maps the Nyquist interval $-\pi < \omega < \pi$ onto the real axis $-\infty < w < \infty$. The famous RBJ Cookbook [1] provides closed form expressions based on the BLT for the coefficients for some standard filter types. The poles determine the characteristic frequency ω_0 in radians per sample and resonance Q , whereas the zeros determine the filter type (lowpass, high-pass, bandpass, etc.). The pole coefficients are:

$$a_1 = \frac{-2 \cos \omega_0}{1 + \alpha}, \quad a_2 = \frac{1 - \alpha}{1 + \alpha} \quad (6)$$

with the abbreviations $\alpha = q \sin \omega_0$ and $q = 1/(2Q)$. For the various filter types, the numerator coefficients are:

$$\text{lowpass: } b_0 = b_2 = (1 + a_1 + a_2)/4, \quad b_1 = 2b_0$$

$$\text{high-pass: } b_0 = b_2 = (1 - a_1 + a_2)/4, \quad b_1 = -2b_0$$

$$\text{bandpass: } b_0 = -b_2 = (1 - a_2)/2, \quad b_1 = 0$$

$$\text{allpass: } b_0 = a_2, \quad b_1 = a_1, \quad b_2 = 1$$

One problem with the BLT is frequency warping towards the Nyquist frequency. For a lowpass filter with a resonant peak the result is a steeper than nominal (12 dB/octave) falloff for high cutoff frequencies, and a narrowing of the resonant peak. Similar effects are present for other standard filter types.

Oversampling provides remedy but at a high CPU cost.

3.2 Impulse Invariance

A continuous-time filter transfer function may be expanded in partial fractions as

$$H_c(s) = \sum_k \frac{A_k}{s - s_k}, \quad (7)$$

which has an impulse response

$$h_c(t) = \sum_k A_k e^{s_k t}. \quad (8)$$

for $t \geq 0$ and zero otherwise. Impulse invariance requires the discrete impulse response h_n to coincide at the sampling times $t = n$,

$$h_n = h_c(n) \quad (9)$$

hence it follows that

$$H(z) = \sum_k \frac{A_k}{1 - e^{s_k} z^{-1}} \quad (10)$$

Equation (10) states that poles are mapped from the analog domain to the digital domain as $z_k = e^{s_k}$. The mapping of zeros of $H(z)$ does not have an intuitive meaning. In fact we will not use the mapped zeros in the following but will only keep the poles.

The analog second order filter prototype has a denominator $s^2 + 2q\omega_0 s + \omega_0^2$ with poles at

$$s_{1,2} = \begin{cases} -q\omega_0 \pm i\sqrt{1 - q^2} \omega_0 & \text{if } q \leq 1 \\ -q\omega_0 \pm \sqrt{q^2 - 1} \omega_0 & \text{if } q > 1 \end{cases} \quad (11)$$

Mapping the poles to the digital domain yields

$$\begin{aligned} a_1 = -(z_1 + z_2) &= \begin{cases} -2e^{-q\omega_0} \cos\left(\sqrt{1 - q^2} \omega_0\right) & \text{if } q \leq 1 \\ -2e^{-q\omega_0} \cosh\left(\sqrt{q^2 - 1} \omega_0\right) & \text{if } q > 1 \end{cases} \\ a_2 = z_1 z_2 &= e^{-2q\omega_0} \end{aligned} \quad (12)$$

We will use these expressions in sections 4 and 5.

3.3 Orfanidis Peaking EQ

Orfanidis [2] proposed a new type of second-order digital parametric equalizer whose frequency response matches closely that of its analog counterpart throughout the Nyquist interval and does not suffer from the prewarping effect of the bilinear transformation near the Nyquist frequency. This is achieved by modification of the analog prototype so that its magnitude response at $s = \infty$ is different from unity. In summary, given the set of digital filter specifications ω_0 (center frequency), $\Delta\omega$ (bandwidth), G_0 (DC gain), G_1 (Nyquist gain), G (peak gain), G_B (gain at $\omega \approx \omega_0 \pm \Delta\omega/2$), use

$$\Omega_0 = \tan\left(\frac{\omega_0}{2}\right), \quad \Delta\Omega = \left(1 + \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}} \sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}} \Omega_0^2\right) \tan\left(\frac{\Delta\omega}{2}\right) \quad (13)$$

to calculate the prewarped analog frequencies. Then, use

$$\begin{aligned} W^2 &= \sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}} \Omega_0^2 \\ C &= (\Delta\Omega)^2 |G_B^2 - G_1^2| - 2W^2 \left(|G_B^2 - G_0G_1| - \sqrt{(G_B^2 - G_0^2)(G_B^2 - G_1^2)} \right) \\ D &= 2W^2 \left(|G^2 - G_0G_1| - \sqrt{(G^2 - G_0^2)(G^2 - G_1^2)} \right) \end{aligned} \quad (14)$$

to calculate the parameters A, B ,

$$A = \sqrt{\frac{C + D}{|G^2 - G_B^2|}}, \quad B = \sqrt{\frac{G^2C + G_B^2D}{|G^2 - G_B^2|}} \quad (15)$$

from which the digital filter coefficients

$$\begin{aligned} b_0 &= \frac{G_1 + G_0W^2 + B}{1 + W^2 + A} & a_1 &= -2 \frac{1 - W^2}{1 + W^2 + A} \\ b_1 &= -2 \frac{G_1 - G_0W^2}{1 + W^2 + A} & a_2 &= \frac{1 + W^2 - A}{1 + W^2 + A} \\ b_2 &= \frac{G_1 + G_0W^2 - B}{1 + W^2 + A} & & \end{aligned} \quad (16)$$

are determined.

If we consider the prototype

$$H_{\text{EQ}}(s) = \frac{\omega_0^2 + s\omega_0\sqrt{G}/Q + s^2}{\omega_0^2 + s\omega_0/(\sqrt{G}Q) + s^2} \quad (17)$$

we may restrict the degrees of freedom by setting

$$G_0 = 1, \quad \Delta\omega = \omega_0/Q, \quad G_B = \sqrt{G} \quad (18)$$

Note that the arguments in the square roots in the above equations require that

$$\begin{aligned} G_0 < G_1 < G_B < G & \text{ (boost)} \\ G_0 > G_1 > G_B > G & \text{ (cut)} \end{aligned} \quad (19)$$

In practical terms this means that the center frequency must be chosen at least half a bandwidth below the Nyquist frequency: $f_0 < (1 - 1/(2Q))f_{\text{Ny}}$.

3.4 Massberg

Massberg [3] also used a modified prototype to obtain a prescribed magnitude response at Nyquist frequency. He started with a shelving filter in the continuous domain to derive a matched discrete lowpass filter.

Specify the cutoff frequency ω_0 and resonance Q for a lowpass. Calculate the gain at Nyquist frequency

$$g_1 = \frac{1}{\sqrt{[1 - (\pi/\omega_0)^2]^2 + [\pi/(Q\omega_0)]^2}} \quad (20)$$

Calculate Ω_s , depending on the value of Q :

$$\begin{aligned} Q > \sqrt{1/2} : & & Q \leq \sqrt{1/2} : \\ g_r = 2Q^2/\sqrt{4Q^2 - 1} & & \omega_m = \omega_0 \sqrt{1 - \frac{1}{2Q^2} + \sqrt{\frac{1 - 4Q^2}{4Q^4} + \frac{1}{g_1}}} \\ \omega_r = \omega_0 \sqrt{1 - 1/(2Q^2)} & & \Omega_m = \tan(\omega_m/2) \\ \Omega_r = \tan(\omega_r/2) & & \Omega_s = \frac{1}{2}\omega_0(1 - g_1^2)^{1/4} \\ \Omega_s = \Omega_r \left(\frac{g_r^2 - g_1^2}{g_r^2 - 1} \right)^{1/4} & & \Omega_s = \min(\Omega_s, \Omega_m) \end{aligned} \quad (21)$$

Calculate the frequencies ω , gains g and Q s for zeros and poles, respectively:

$$\begin{aligned}
\omega_z &= 2 \arctan(\Omega_s/\sqrt{g_1}) & \omega_p &= 2 \arctan \Omega_s \\
g_z &= \frac{1}{\sqrt{[1 - (\omega_z/\omega_0)^2]^2 + [\omega_z/(Q\omega_0)]^2}} & g_p &= \frac{1}{\sqrt{[1 - (\omega_p/\omega_0)^2]^2 + [\omega_p/(Q\omega_0)]^2}} \\
Q_z &= \sqrt{\frac{g_1^2(g_p^2 - g_z^2)}{g_z^2(g_1 + g_p^2)(g_1 - 1)^2}} & Q_p &= \sqrt{\frac{g_1(g_p^2 - g_z^2)}{(g_1 + g_z^2)(g_1 - 1)^2}} \quad (22)
\end{aligned}$$

Proceed as follows:

$$\begin{aligned}
\gamma_0 &= \Omega_s^2 + \Omega_s/Q_p + 1 \\
\beta_0 &= \Omega_s^2 + \sqrt{g_1}\Omega_s/Q_z + g_1 \\
\beta_1 &= 2(\Omega_s^2 - g_1) & \alpha_1 &= 2(\Omega_s^2 - 1) \\
\beta_2 &= \Omega_s^2 - \sqrt{g_1}\Omega_s/Q_z + g_1 & \alpha_2 &= \Omega_s^2 - \Omega_s/Q_p + 1 \\
b_0 &= \alpha_0/\gamma_0 \\
b_1 &= \alpha_1/\gamma_0 & a_1 &= \beta_1/\gamma_0 \\
b_2 &= \alpha_2/\gamma_0 & a_2 &= \beta_2/\gamma_0 \quad (23)
\end{aligned}$$

Note: the a s and b s from the original paper have been swapped for consistency with the rest of this article.

4 Custom Matched Biquad

In this section we present an alternative scheme to directly match the filter's magnitude response to a given analog prototype, recall equation (2). Although $H(z)$ is defined on the complex plane, the phase and magnitude response to a sine wave excitation with frequency ω (radians per sample) is actually found on the unit circle $z = e^{i\omega}$, with $z = 1$ representing DC and $z = -1$ or $\omega = \pi$ the Nyquist limit.

From eq. (2) the magnitude response $|H(z)|$ is given by

$$\begin{aligned}
|H(z)|^2 &= \frac{[(b_0 + b_2) \cos \omega + b_1]^2 + [(b_0 - b_2) \sin \omega]^2}{[(1 + a_2) \cos \omega + a_1]^2 + [(1 - a_2) \sin \omega]^2} \\
&= \frac{b_1^2 + (b_0 - b_2)^2 + 2b_1(b_0 + b_2) \cos \omega + 4b_0b_2 \cos^2 \omega}{a_1^2 + (1 - a_2)^2 + 2a_1(1 + a_2) \cos \omega + 4a_2 \cos^2 \omega} \quad (24)
\end{aligned}$$

Equation (24) may be cast in a form which simplifies the subsequent derivation, and which is moreover favorable with regard to numerical evaluation:

$$|H(z)|^2 = \frac{B_0\phi_0 + B_1\phi_1 + B_2\phi_2}{A_0\phi_0 + A_1\phi_1 + A_2\phi_2} \quad (25)$$

with

$$\phi_0 = 1 - \sin^2(\omega/2), \quad \phi_1 = \sin^2(\omega/2), \quad \phi_2 = 4\phi_0\phi_1 \quad (26)$$

and

$$\begin{aligned} A_0 &= (1 + a_1 + a_2)^2 & B_0 &= (b_0 + b_1 + b_2)^2 \\ A_1 &= (1 - a_1 + a_2)^2 & B_1 &= (b_0 - b_1 + b_2)^2 \\ A_2 &= -4a_2 & B_2 &= -4b_0b_2 \end{aligned} \quad (27)$$

One possible way to proceed would be to match $|H(z)|^2$ in equation (25) at five points and solve the resulting set of linear equations for A_i and B_i . Then eq. (27) may be inverted to yield the filter coefficients a_i and b_i . We will not take this route, however.

In what follows, we will take the poles from impulse invariance, eq. (12), to avoid peak narrowing towards Nyquist. Furthermore we will derive custom numerator coefficients to avoid a too steep falloff at high frequencies. In particular, we will avoid zeros at Nyquist.

The numerator coefficients will be derived by matching the low-frequency behavior and the magnitude response at the cutoff or center frequency ω_0 . No special requirement is imposed at the Nyquist frequency.

We note that once the quantities B_0, B_1, B_2 have been determined, the coefficients b_0, b_1, b_2 may be computed by solving equation (27). The solution is not unique, however, unless we impose a further condition regarding the location of zeros. We require the zeros to lie within the unit circle for a minimum phase filter. This is equivalent to

$$b_0 > |b_2|, \quad b_0 + b_2 > |b_1|. \quad (28)$$

It is not difficult to derive the solution for the b s:

$$\begin{aligned} b_0 &= \frac{1}{2} \left(W + \sqrt{W^2 + B_2} \right), \quad \text{with} \quad W = \frac{1}{2} \left(\sqrt{B_0} + \sqrt{B_1} \right) \\ b_1 &= \frac{1}{2} \left(\sqrt{B_0} - \sqrt{B_1} \right) \\ b_2 &= -B_2 / (4b_0) \end{aligned} \quad (29)$$

4.1 Matched Lowpass Filter

Consider the analog prototype with cutoff frequency ω_0 and resonance Q :

$$H_{\text{LP}}(s) = \frac{\omega_0^2}{\omega_0^2 + s\omega_0/Q + s^2} \quad (30)$$

It turns out that the lowpass filter can be matched quite well with a single zero in the digital domain, so we set $b_2 = 0$. The remaining coefficients b_0 and b_1 are determined by the following requirements:

1. Unity gain at DC: $H(z = 1) = 1$
2. Match the prototype magnitude response at cutoff frequency:
 $|H(z)|_{\omega=\omega_0} = |H_{\text{LP}}(s = i\omega_0)| = Q$

The two requirements lead to the following set of equations:

$$\begin{aligned} B_0 &= A_0 \\ B_0\phi_0 + B_1\phi_1 &= (A_0\phi_0 + A_1\phi_1 + A_2\phi_2) Q^2 =: R_1 \end{aligned} \quad (31)$$

The solution is easily obtained,

$$B_1 = (R_1 - B_0\phi_0)/\phi_1 \quad (32)$$

and from there

$$\begin{aligned} b_0 &= \frac{1}{2} \left(\sqrt{B_0} + \sqrt{B_1} \right) \\ b_1 &= \sqrt{B_0} - b_0 \end{aligned} \quad (33)$$

The first square root in the expression for b_0 is simply

$$\sqrt{B_0} = 1 + a_1 + a_2 \quad (34)$$

The other root cannot be simplified that way. Unfortunately, its computation requires double precision.

4.2 Matched Highpass Filter

The analog highpass prototype reads:

$$H_{\text{HP}}(s) = \frac{s^2}{\omega_0^2 + s\omega_0/Q + s^2} \quad (35)$$

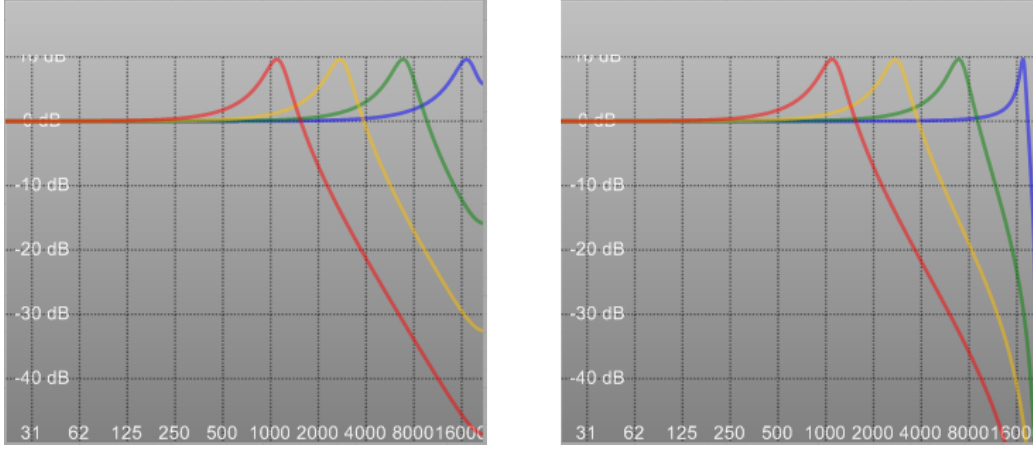


Figure 2: Magnitude response of various lowpass filters. Left: matched lowpass. Right: BLT lowpass. In all cases $Q = 3$. In both figures: Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

We want to reproduce the low frequency attenuation at 12 dB/octave in the digital domain, which means that our digital filter has a double zero at DC. Furthermore, we require that the prototype be matched at the cutoff frequency ω_0 , so the requirements are:

1. Double zero at $z = 1$
2. Match prototype at cutoff frequency:
 $|H(z)|_{\omega=\omega_0} = |H_{HP}(s = i\omega_0)| = Q$

The first requirement implies $b_1 = -2b_0$, $b_2 = b_0$. The second requirement results in the following expression for b_0 :

$$b_0 = \frac{\sqrt{A_0\phi_0 + A_1\phi_1 + A_2\phi_2}}{4\phi_1} Q \quad (36)$$

Although this result looks simple, again its computation requires double precision.

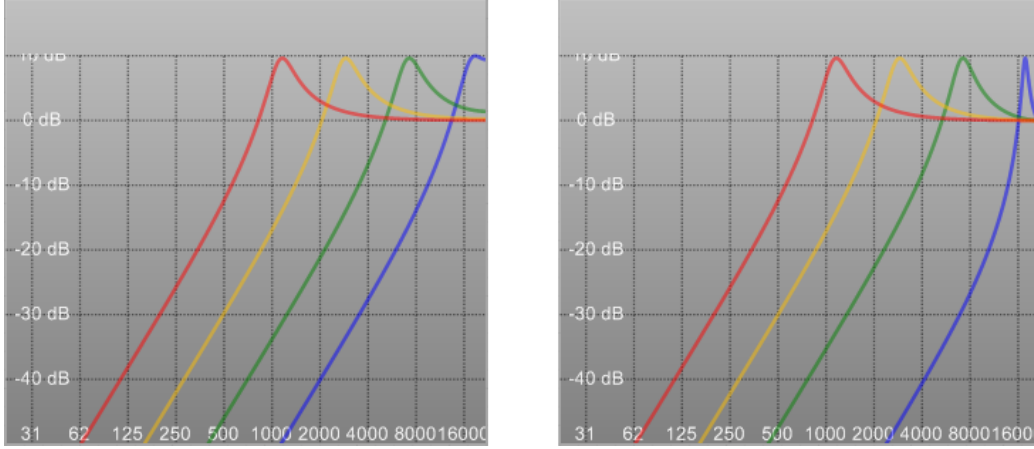


Figure 3: Magnitude response of various highpass filters. Left: matched highpass. Right: BLT highpass. In all cases $Q = 3$. In both figures: Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

4.3 Matched Bandpass Filter

The analog prototype for a bandpass filter with unity gain at the center frequency ω_0 is given by

$$H_{\text{BP}}(s) = \frac{s\omega_0/Q}{\omega_0^2 + s\omega_0/Q + s^2} \quad (37)$$

Low frequencies $\omega \ll \omega_0$ are attenuated with a 6 dB/octave slope, which implies for our digital filter a single zero at $z = 1$. As in the previous cases, we want the digital and analog filters to match at ω_0 . However, in this case we have two remaining degrees of freedom so we have the following requirements:

1. Single zero at DC: $b_0 + b_1 + b_2 = 0$.
2. Maximum at ω_0 : $\frac{d}{d\omega}|H(z)|^2|_{\omega=\omega_0} = 0$.
3. Unity gain at ω_0 : $|H(z)|^2|_{\omega=\omega_0} = 1$

The first requirement yields $B_0 = 0$. The second and third requirements imply

$$\frac{B_1 + 4(\phi_0 - \phi_1)B_2}{-A_0 + A_1 + 4(\phi_0 - \phi_1)A_2} = \frac{B_1\phi_1 + B_2\phi_2}{A_0\phi_0 + A_1\phi_1 + A_2\phi_2} = 1. \quad (38)$$

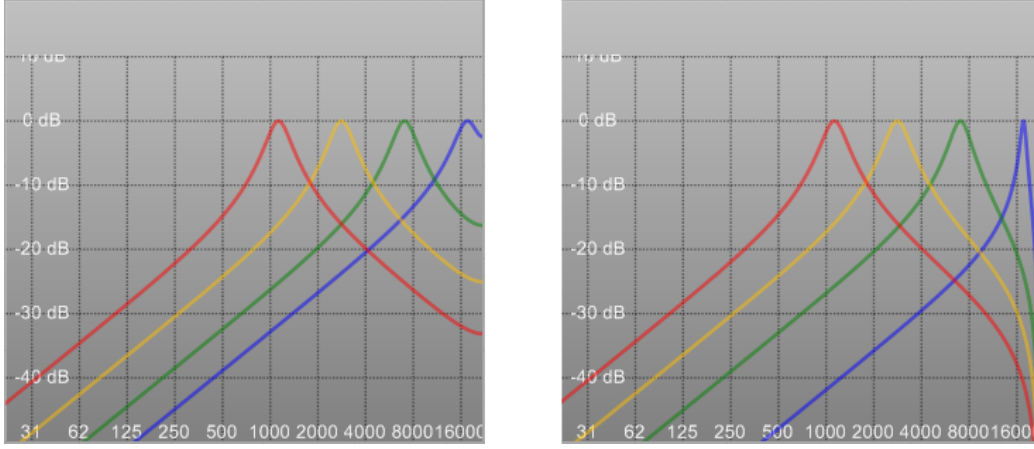


Figure 4: Magnitude response of various bandpass filters. Left: matched bandpass. Right: BLT bandpass. In all cases $Q = 3$. In both figures: Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

Hence we obtain a system of linear equations for the unknowns B_1 and B_2 :

$$\begin{aligned} B_1\phi_1 + B_2\phi_2 &= A_0\phi_0 + A_1\phi_1 + A_2\phi_2 =: R_1 \\ B_1 + 4(\phi_0 - \phi_1)B_2 &= -A_0 + A_1 + 4(\phi_0 - \phi_1)A_2 =: R_2 \end{aligned} \quad (39)$$

The solution is

$$B_2 = \frac{R_1 - R_2\phi_1}{4\phi_1^2}, \quad B_1 = R_2 + 4(\phi_1 - \phi_0)B_2. \quad (40)$$

From there,

$$\begin{aligned} b_1 &= -\frac{1}{2}\sqrt{B_1} \\ b_0 &= \frac{1}{2}\left(\sqrt{B_2 + b_1^2} - b_1\right) \\ b_2 &= -b_0 - b_1 \end{aligned} \quad (41)$$

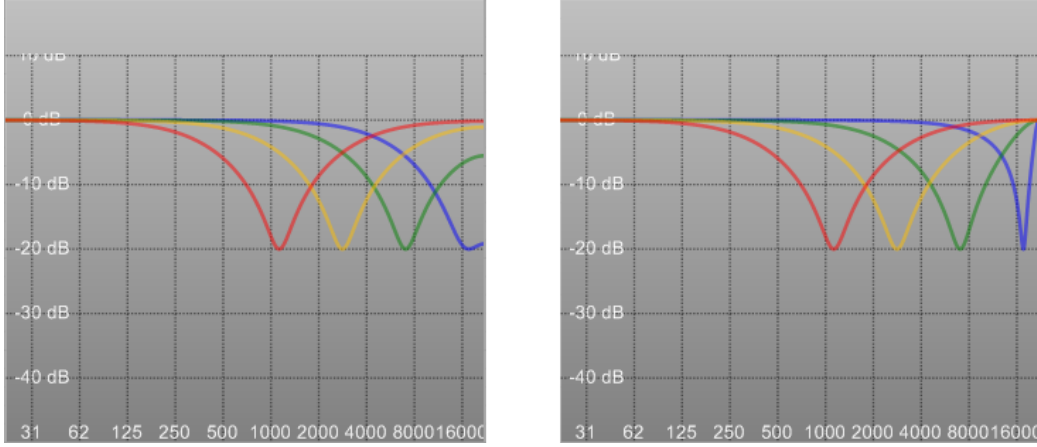


Figure 5: Magnitude response of various peaking EQ filters. Left: matched filter. Right: BLT filter. In all cases $Q = 1$ and $G = 0.1$ (or -20 dB). In both figures: Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

4.4 Matched Peaking EQ Filter

The analog prototype for a peaking EQ filter is

$$H_{\text{EQ}}(s) = \frac{\omega_0^2 + s\omega_0\sqrt{G}/Q + s^2}{\omega_0^2 + s\omega_0/(\sqrt{G}Q) + s^2} \quad (42)$$

This transfer function has unity gain for $s = 0$ and $s = \infty$. At the center frequency ω_0 , the magnitude response has a maximum or minimum with a value G . For the digital counterpart, we require unity gain at DC and matching at ω_0 :

1. Unity gain at DC: $H(z = 1) = 1$
2. Match prototype at center frequency: $|H(z)|_{\omega=\omega_0} = G$
3. Maximum or minimum at center frequency: $\frac{d}{d\omega}|H(z)|^2|_{\omega=\omega_0} = 0$

Condition 1. yields $B_0 = A_0$. Conditions 2. and 3. imply

$$\frac{-B_0 + B_1 + 4(\phi_0 - \phi_1)B_2}{-A_0 + A_1 + 4(\phi_0 - \phi_1)A_2} = \frac{B_0\phi_0 + B_1\phi_1 + B_2\phi_2}{A_0\phi_0 + A_1\phi_1 + A_2\phi_2} = G^2. \quad (43)$$

Hence we obtain a system of linear equations for the unknowns B_1 and B_2 :

$$\begin{aligned} B_0\phi_0 + B_1\phi_1 + B_2\phi_2 &= (A_0\phi_0 + A_1\phi_1 + A_2\phi_2)G^2 =: R_1 \\ -B_0 + B_1 + 4(\phi_0 - \phi_1)B_2 &= (-A_0 + A_1 + 4(\phi_0 - \phi_1)A_2)G^2 =: R_2 \end{aligned} \quad (44)$$

The solution is

$$B_2 = \frac{R_1 - R_2\phi_1 - B_0}{4\phi_1^2}, \quad B_1 = R_2 + B_0 + 4(\phi_1 - \phi_0)B_2. \quad (45)$$

From there, we may use the solution in equation (29).

5 Even Simpler Fits

For musical applications the above expressions may be too complicated to allow fast computation of modulated filter coefficients. Another drawback is the sensitivity of the filter characteristics on numerical accuracy. Both issues will be addressed in this section, resulting in simpler and more robust closed form expressions for the filter coefficients. The price is a slightly worse match of the analog prototype, albeit still superior to the BLT.

5.1 Lowpass

Criteria for the transfer function are: (i) unity at DC, (ii) match magnitude at Nyquist, (iii) set $b_2 = 0$:

$$\begin{aligned} b_0 + b_1 &= 1 + a_1 + a_2 =: r_0 \\ b_0 - b_1 &= \frac{(1 - a_1 + a_2)f_0^2}{\sqrt{(1 - f_0^2)^2 + f_0^2/Q^2}} =: r_1 \end{aligned} \quad (46)$$

The solution is:

$$b_0 = (r_0 + r_1)/2, \quad b_1 = r_0 - b_0 \quad (47)$$

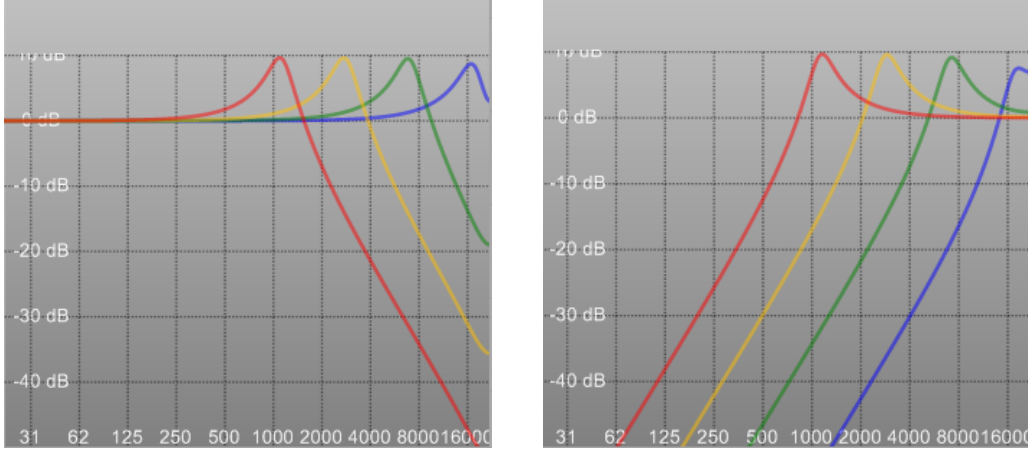


Figure 6: Magnitude response of various filters matched at DC and at the Nyquist frequency. Left: matched lowpass. Right: matched highpass. In all cases $Q = 3$. In both figures: Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

5.2 Highpass

Criteria are: (i) double zero at DC, (ii) match at Nyquist:

$$\begin{aligned}
 b_0 + b_1 + b_2 &= 0, \\
 b_0 - b_2 &= 0 \\
 b_0 - b_1 + b_2 &= \frac{1 - a_1 + a_2}{\sqrt{(1 - f_0^2)^2 + f_0^2/Q^2}} =: r_1
 \end{aligned} \tag{48}$$

The solution is:

$$b_0 = r_1/4, \quad b_1 = -2b_0, \quad b_2 = b_0. \tag{49}$$

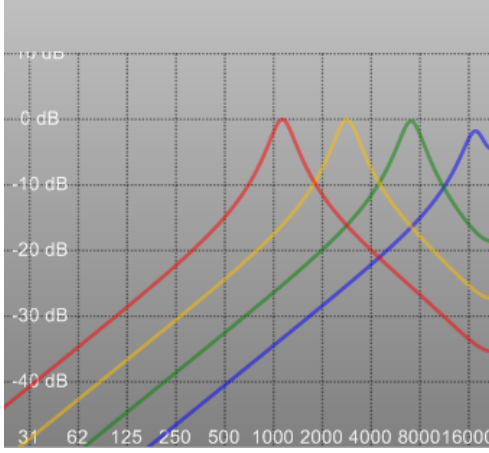


Figure 7: Magnitude response of various bandpass filters matched at DC and at the Nyquist frequency. In all cases $Q = 3$. Red: $f_0 = 0.0512$. Yellow: $f_0 = 0.128$. Green: $f_0 = 0.32$. Blue: $f_0 = 0.8$.

5.3 Bandpass

Criteria are: (i) single zero at DC, (ii) match slope at DC, (iii) match magnitude at Nyquist:

$$\begin{aligned}
 b_0 + b_1 + b_2 &= 0, \\
 b_0 - b_2 &= \frac{1 + a_1 + a_2}{\pi f_0 Q} =: r_0 \\
 b_0 - b_1 + b_2 &= \frac{(1 - a_1 + a_2)f_0/Q}{\sqrt{(1 - f_0^2)^2 + f_0^2/Q^2}} =: r_1
 \end{aligned} \tag{50}$$

The solution is:

$$b_1 = -r_1/2, \quad b_0 = (r_0 - b_1)/2, \quad b_2 = -b_0 - b_1 \tag{51}$$

6 Conclusion

Digital biquad filters may be matched to have a similar magnitude response as their analog counterparts. The five degrees of freedom may be reduced by choosing the filter poles according to impulse invariance. The remaining

filter coefficients may be determined by matching the magnitude response to that of the corresponding analog filter. The result is a uniform filter response without the notorious cramping near the Nyquist frequency as is typical of bilinear transformation designs. The matched coefficients are given in closed form, and are simpler than in previous treatments, thus being suitable for modulation where fast coefficients updating is needed.

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